

REAL QUASI BIHILBERT SPACE

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ABSTRACT. In this paper, we introduce the notion of quasi-inner product and conjugate quasi-inner product mappings on a real vector space. We then define a real quasi *bi*Hilbert space and study several properties of the space.

Keywords and phrases: Quasi-inner product; conjugate quasi-inner product; real quasi *bi*Hilbert space; asymmetric norm; and conjugate asymmetric norm.

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1. INTRODUCTION

A quasi-metric on a set X is a non-negative real-valued bifunction $p(\cdot, \cdot)$ on the product $X \times X$ such that, for all $x, y, z \in X$

- (1) $p(x, y) \geq 0$ and $p(x, x) = 0$;
- (2) $p(x, z) \leq p(x, y) + p(y, z)$;
- (3) $p(x, y) = p(y, x) = 0 \Rightarrow x = y$.

Given a quasi metric p on X , the pair (X, p) is called a quasi metric space. We call two quasi metrics p and q on X conjugates if $q(x, y) = p(y, x)$, for all $x, y \in X$.

In [2,3], let X be real vector space. an asymmetric norm is a positive sublinear functional $\|\cdot\|$ on a real vector space X which satisfies, for arbitrary $x, y \in X$

- (A1) $\|x\| \geq 0$;
- (A2) $\|\alpha x\| = \alpha \|x\|$ $\alpha \geq 0$;
- (A3) $\|x + y\| \leq \|x\| + \|y\|$;
- (A4) $\|x\| = \|-x\| = 0 \Rightarrow x = 0$.

The conjugate asymmetric norm $|\cdot|$ is given by $|x| = \|-x\|$. X together with $\|\cdot\|$ forms a real asymmetric normed space.

In case X is a real vector space, the asymmetric norm and conjugate asymmetric norm induce a quasi metric and conjugate quasi metric, respectively, by $p(x, y) = \|x - y\|$ and $q(x, y) = |x - y|$.

Examples of real asymmetric normed spaces:

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- (1) $\|\cdot\|^* : \mathbb{R} \rightarrow \mathbb{R}$ given by $\|x\|^* := \max\{x, 0\}$, is an asymmetric norm on $\mathbb{R}$ and so $(\mathbb{R}, \|\cdot\|^*)$ is an asymmetric normed space with its conjugate defined by $|x|^* := \max\{-x, 0\}$.
- (2) $(\mathbb{R}^n, \|\cdot\|_2)$ is an asymmetric normed space, where $\|\cdot\|_2$ is given by $\|x\|_2 := (\sum_{i=1}^n (\max\{x_i, 0\})^2)^{\frac{1}{2}}$ and its conjugate $|x|_2 := (\sum_{i=1}^n (\max\{-x_i, 0\})^2)^{\frac{1}{2}}$.
- (3) $(l_p, \|\cdot\|_p)$ is an asymmetric normed space, where $\|\cdot\|_p$ is given by $\|x\|_p := (\sum_{i=1}^\infty (\max\{x_i, 0\})^p)^{\frac{1}{p}}$ and its conjugate $|x|_p := (\sum_{i=1}^\infty (\max\{-x_i, 0\})^p)^{\frac{1}{p}}$.
- (4) $(C[a, b], \|\cdot\|_\infty)$ is an asymmetric normed space, where $\|\cdot\|_\infty$ is defined by $\|f\|_\infty := \sup_{x \in [a, b]} \max\{f(x), 0\}$ and its conjugate $|f|_p := \sup_{x \in [a, b]} \max\{-f(x), 0\}$.

The purpose of this paper is to introduce the notion of real quasi biHilbert space and then study some of its properties.

2. MAIN RESULTS

2.1 Real quasi bi-inner product space

Definition 2.1 Let X be a real vector space. A quasi inner product $\langle \cdot, \cdot \rangle^+ : X \times X \rightarrow \mathbb{R}$ is a bifunction satisfying the following conditions: for arbitrary $x, y, z \in X$,

- (Q1) $\langle x, x \rangle^+ \geq 0$,
- (Q2) $\langle x, y \rangle^+ = \langle y, x \rangle^+$,
- (Q3) $\langle \alpha x, y \rangle^+ = \langle x, \alpha y \rangle^+ = \alpha \langle x, y \rangle^+$, $\alpha \geq 0$,
- (Q4) $\langle x + y, z \rangle^+ \geq \langle x, z \rangle^+ + \langle y, z \rangle^+$,
- (Q5) $\langle x, x \rangle^+ = \langle -x, -x \rangle^+ = 0 \Rightarrow x = 0$.

The conjugate quasi inner product is given by $\langle x, x \rangle^- = \langle -x, -x \rangle^+$. The triplet $(X, \langle \cdot, \cdot \rangle^+, \langle \cdot, \cdot \rangle^-)$ is defined to be a real quasi bi-inner product space.

2.2 Examples of real quasi bi-inner product spaces

Let $x, y \in X$, then

- (1) $(X = \mathbb{R}, \langle \cdot, \cdot \rangle^+, \langle \cdot, \cdot \rangle^-)$ is a quasi bi-inner product space, where $\langle \cdot, \cdot \rangle^+$ is defined by

$$\langle x, y \rangle^+ = \begin{cases} \min\{x, 0\} \min\{y, 0\}, & \text{sgn}(x) = \text{sgn}(y) \\ xy, & \text{sgn}(x) \neq \text{sgn}(y), \end{cases}$$

and the conjugate is

$$\langle x, y \rangle^- = \begin{cases} \min\{-x, 0\} \min\{-y, 0\}, & \text{sgn}(-x) = \text{sgn}(-y) \\ (-x)(-y), & \text{sgn}(-x) \neq \text{sgn}(-y). \end{cases}$$

- (2) $(X = \mathbb{R}^n, \langle \cdot, \cdot \rangle^+, \langle \cdot, \cdot \rangle^-)$ is a quasi bi-inner product space, where $\langle \cdot, \cdot \rangle^+$ is defined by

$$\langle x, y \rangle^+ = \begin{cases} \sum_{i=1}^n \min\{x_i, 0\} \min\{y_i, 0\}, & \text{sgn}(x_i) = \text{sgn}(y_i) \\ \sum_{i=1}^n x_i y_i, & \text{sgn}(x_i) \neq \text{sgn}(y_i), \end{cases}$$

and the conjugate is

$$\langle x, y \rangle^- = \begin{cases} \sum_{i=1}^n \min\{-x_i, 0\} \min\{-y_i, 0\}, & \text{sgn}(-x_i) = \text{sgn}(-y_i) \\ \sum_{i=1}^n (-x_i)(-y_i), & \text{sgn}(-x_i) \neq \text{sgn}(-y_i). \end{cases}$$

- (3) $(X = l_2, \langle \cdot, \cdot \rangle^+, \langle \cdot, \cdot \rangle^-)$ is a quasi bi-inner product space, where $\langle \cdot, \cdot \rangle^+$ is defined by

$$\langle x, y \rangle^+ = \begin{cases} \sum_{i=1}^{\infty} \min\{x_i, 0\} \min\{y_i, 0\}, & \text{sgn}(x_i) = \text{sgn}(y_i) \\ \sum_{i=1}^{\infty} x_i y_i, & \text{sgn}(x_i) \neq \text{sgn}(y_i), \end{cases}$$

and the conjugate is

$$\langle x, y \rangle^- = \begin{cases} \sum_{i=1}^{\infty} \min\{-x_i, 0\} \min\{-y_i, 0\}, & \text{sgn}(-x_i) = \text{sgn}(-y_i) \\ \sum_{i=1}^{\infty} (-x_i)(-y_i), & \text{sgn}(-x_i) \neq \text{sgn}(-y_i). \end{cases}$$

- (4) $(X = C[0, 1], \langle \cdot, \cdot \rangle^+, \langle \cdot, \cdot \rangle^-)$ is a quasi bi-inner product space, where $\langle \cdot, \cdot \rangle^+$ is defined by

$$\langle f, g \rangle^+ = \begin{cases} \int_0^1 \min\{f(x), 0\} \min\{g(x), 0\} dx, & \text{sgn}(f) = \text{sgn}(g) \\ \int_0^1 f(x)g(x) dx, & \text{sgn}(f) \neq \text{sgn}(g). \end{cases}$$

and the conjugate is

$$\langle f, g \rangle^- = \begin{cases} \int_0^1 \min\{-f(x), 0\} \min\{-g(x), 0\} dx, & \text{sgn}(-f) = \text{sgn}(-g) \\ \int_0^1 (-f(x))(-g(x)) dx, & \text{sgn}(-f) \neq \text{sgn}(-g). \end{cases}$$

2.3 Real quasi *bi*Hilbert space

Theorem 2.1 (Cauchy-Schwartz type inequality). Let X be a real quasi *bi*-inner product space. Then

$$|\langle x, y \rangle^-|^2 \leq \langle x, x \rangle^- \langle y, y \rangle^-, \quad (1)$$

for all $x, y \in X$.

Proof. Let $x, y \in X$. Then for arbitrary $t \in [0, \infty)$, we have

$$\langle tx + y, tx + y \rangle^- \geq \langle tx, tx \rangle^- + 2\langle tx, y \rangle^- + \langle y, y \rangle^-. \quad (2)$$

Interchanging the role of tx and $tx + y$ in (2), we get

$$\begin{aligned} 0 \leq \langle tx + y, tx + y \rangle^- &\leq t^2 \langle x, x \rangle^- - 2t \langle x, y \rangle^- - \langle y, y \rangle^- \\ &\leq t^2 \langle x, x \rangle^- + 2t |\langle x, y \rangle^-| + \langle y, y \rangle^-, \end{aligned}$$

so that

$$0 \leq at^2 + bt + c,$$

where $a = \langle x, x \rangle^-$, $b = 2|\langle x, y \rangle^-|$ and $c = \langle y, y \rangle^-$. Therefore, by the theory of quadratic inequality, the discriminant of the the last inequality above is nonpositive. Thus we get, $b^2 \leq 4ac$ which gives (1), as desired.

Theorem 2.2 Let $(X, \langle \cdot, \cdot \rangle^+, \langle \cdot, \cdot \rangle^-)$ be a real quasi bi-inner product space and let $x \in X$. Then the functions $\|\cdot\|$ and $|\cdot|$ defined by $\|x\| = \sqrt{\langle x, x \rangle^-}$ and $|x| = \sqrt{\langle x, x \rangle^+}$ give asymmetric norm and conjugate asymmetric norm, respectively.

Proof. To prove this, it suffices to show that the given functions satisfy (A1) – (A4) conditions above. By (Q1), (Q3) and (Q5), we get trivially (A1), (A2) and (A4), respectively. For (A3), let $x, y \in X$, then by (Q4) and from the Cauchy-Schwartz type inequality, we obtain

$$\begin{aligned} -\|x + y\|^2 &= -\langle x + y, x + y \rangle^- \\ &\leq -\langle x, x \rangle^- - 2\langle x, y \rangle^- - \langle y, y \rangle^-, \\ &\leq -\|x\|^2 + 2|\langle x, y \rangle^-| - \|y\|^2, \\ &\leq -\|x\|^2 + 2\|x\|\|y\| - \|y\|^2. \end{aligned}$$

This implies that

$$\|x + y\|^2 \geq \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 = (\|x\| - \|y\|)^2. \quad (3)$$

Interchanging the role of x and $x + y$ in (3), we get

$$\|x + y\| \leq \|x\| + \|y\|, \text{ as required in (A3).}$$

Hence, in a similar way, we see that both $\|\cdot\|$ and $|\cdot|$ are well defined as claimed.

Theorem 2.3 Let X be a real vector space. Then $\langle \cdot, \cdot \rangle^+$ and $\langle \cdot, \cdot \rangle^-$ are continuous bifunctions on $X \times X$.

Proof. We now begin with some analysis, that is, for $x, x_0, a \in X$, we have

$$\langle x_0, x - a \rangle^- \geq \langle x_0, x \rangle^- + \langle x_0, -a \rangle^-. \quad (4)$$

Replacing $-a$ by $x - a$ in (4), we have

$$\langle x_0, x - a \rangle^- \leq \langle x_0, -a \rangle^- - \langle x_0, x \rangle^-. \quad (5)$$

Combining (4) and (5), we obtain

$$\langle x_0, x \rangle^- + \langle x_0, -a \rangle^- \leq \langle x_0, x - a \rangle^- \leq \langle x_0, -a \rangle^- - \langle x_0, x \rangle^-.$$

If $\langle x_0, x \rangle^- < 0$, then it implies that

$$|\langle x_0, x - a \rangle^- - \langle x_0, -a \rangle^-| \leq \langle x_0, x \rangle^- \leq |\langle x_0, x \rangle^-|. \quad (6)$$

Else if, $\langle x_0, x \rangle^- \geq 0$, then by (4), we have

$$\langle x_0, x \rangle^- + \langle x_0, x - a \rangle^- \geq \langle x_0, -a \rangle^-.$$

This implies that

$$\langle x_0, x - a \rangle^- \geq \langle x_0, -a \rangle^- - \langle x_0, x \rangle^-. \quad (7)$$

Replacing $x - a$ by $-a$ in (7), we get

$$\langle x_0, x - a \rangle^- \leq \langle x_0, -a \rangle^- + \langle x_0, x \rangle^-. \quad (8)$$

(7) and (8) also implies (6). Therefore, interchanging the role of x and $x - a$ in (6), we get

$$|\langle x_0, x \rangle^- - \langle x_0, -a \rangle^-| \leq |\langle x_0, x - a \rangle^-|, \quad (9)$$

and replacing $-a$ by a in (9), we arrive at

$$|\langle x_0, x \rangle^- - \langle x_0, a \rangle^-| \leq |\langle x_0, x - a \rangle^-|. \quad (10)$$

So, for arbitrary $x, x_0 \in X$ with $0 \neq x_0$ fixed, let $a \in X$, by the continuity of $\|\cdot\|$, for any $\epsilon > 0$, there exists $\delta > 0$ such that, by

(10) and the Cauchy-Schwartz type inequality

$$\begin{aligned} |\langle x_0, x \rangle^- - \langle x_0, a \rangle^-| &\leq |\langle x_0, x - a \rangle^-|, \\ &\leq \sqrt{\|x_0\| \|x - a\|} < \sqrt{\delta \|x_0\|} = \epsilon, \\ \text{for } \delta &= \frac{\epsilon^2}{\|x_0\|}. \end{aligned}$$

But, for $x_0 = 0$, we are done with the proof. Therefore, the function $\langle \cdot, \cdot \rangle^-$ is continuous with respect to the first argument. Clearly, we see that the continuity also holds with respect to the second variable. Hence, in a similar way, both $\langle \cdot, \cdot \rangle^+$ and $\langle \cdot, \cdot \rangle^-$ are continuous as required.

Definition 2.2 Let $(X, \langle \cdot, \cdot \rangle^+, \langle \cdot, \cdot \rangle^-)$ be a real quasi *bi*-inner product space. A sequence $\{x_n\}$ in X is called Cauchy with respect to $\|\cdot\|$, denoted by $\|\cdot\|$ -Cauchy, (Cauchy with respect to $|\cdot|$, denoted by $|\cdot|$ -Cauchy) if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ ($n'_0 \in \mathbb{N}$) such that $\forall n, m \in \mathbb{N}$, $m \geq n \geq n_0$ ($m \geq n \geq n'_0$) implies

$$\|x_n - x_m\| < \epsilon \text{ (respectively } |x_n - x_m| < \epsilon).$$

It is therefore called *bi*Cauchy if it is both $\|\cdot\|$ -Cauchy and $|\cdot|$ -Cauchy.

Definition 2.3 Let $(X, \langle \cdot, \cdot \rangle^+, \langle \cdot, \cdot \rangle^-)$ be a real quasi *bi*-inner product space. A sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to $\|\cdot\|$, denoted by $\|\cdot\|$ -converges, (with respect to $|\cdot|$, denoted by $|\cdot|$ -converges) if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ ($n'_0 \in \mathbb{N}$) such that $\forall n \geq n_0$ ($\forall n \geq n'_0$) implies

$$\|x_n - x\| < \epsilon \text{ (respectively } |x_n - x| < \epsilon).$$

The sequence is therefore called *biconvergent* if it is both $\|\cdot\|$ -convergent and $|\cdot|$ -convergent.

Definition 2.4 Let $(X, \langle \cdot, \cdot \rangle^+, \langle \cdot, \cdot \rangle^-)$ be a real quasi *bi*-inner product space. If every *bi*Cauchy sequences *biconverges*, then X is called a *bicomplete* real quasi *bi*-inner product space.

Definition 2.5 A *bicomplete* real quasi *bi*-inner product space is called real quasi *bi*Hilbert space.

2.4 Geometric properties of real quasi *bi*Hilbert spaces

Theorem 2.4 (Parallelogram type law). Let X be a real quasi bi -inner product space. For all $x, y \in X$, the parallelogram type inequality holds

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &\geq 2(\|x\|^2 + 2^{-1}(\|y\|^2 + \|y\|^2)) \\ &\quad + 2[\langle x, y \rangle^- + \langle x, -y \rangle^-]. \end{aligned} \quad (11)$$

Proof. The proof of (11) follows by expanding the left hand side to obtain the right hand side.

Next, we prove the following very important inequality.

Theorem 2.5 Let X be a real quasi bi -inner product space. For arbitrary $x, y \in X$, the following inequality holds

$$\langle x, y \rangle^- + \langle x, -y \rangle^- \leq 0 \text{ (similarly } \langle x, y \rangle^+ + \langle x, -y \rangle^+ \leq 0). \quad (12)$$

Proof. Let $\bar{0}$ denotes a zero vector, then for arbitrary $x, y \in X$,

$$\begin{aligned} 0 &= 0\langle x, \bar{0} \rangle^- \\ &= \langle x, 0\bar{0} \rangle^- \text{ (by condition } Q3), \\ &= \langle x, \bar{0} \rangle^- = \langle x, y + (-y) \rangle^-, \\ &\geq \langle x, y \rangle^- + \langle x, -y \rangle^- \text{ (by condition } Q4). \end{aligned}$$

Hence, (12) holds, as required.

Theorem 2.6 Let X be a real quasi bi -inner product space. Then,

$$\langle x, y \rangle^- + \langle x, -y \rangle^- < 0,$$

if and only if

$$\|x + y\|^2 + \|x - y\|^2 < 2(\|x\|^2 + 2^{-1}(\|y\|^2 + \|y\|^2)),$$

for all $x, y \in X$.

Proof. By (11),

$$A \geq B + C, \quad (13)$$

where

$$\begin{aligned} A &= \|x + y\|^2 + \|x - y\|^2, \\ B &= 2(\|x\|^2 + 2^{-1}(\|y\|^2 + \|y\|^2)) \end{aligned}$$

and

$$C = 2[\langle x, y \rangle^- + \langle x, -y \rangle^-].$$

So,

$$\begin{aligned}
& 0 > C, \text{ (by the hypothesis of the theorem)} \\
& \text{equivalently} \\
& -A + A > -B + B + C, \\
& \text{equivalently} \\
& -A > -B \text{ (by (13))} \\
& \text{equivalently} \\
& A < B,
\end{aligned}$$

as desired.

Theorem 2.7 Let X be a real quasi *bi*-inner product space. Then,

$$\langle x, y \rangle^- + \langle x, -y \rangle^- = 0,$$

if and only if

$$\|x + y\|^2 + \|x - y\|^2 \geq 2(\|x\|^2 + 2^{-1}(\|y\|^2 + \|y\|^2)),$$

for all $x, y \in X$.

Proof. ($\Rightarrow$). Let $A = \|x + y\|^2 + \|x - y\|^2$, $B = 2(\|x\|^2 + 2^{-1}(\|y\|^2 + \|y\|^2))$ and $C = 2[\langle x, y \rangle^- + \langle x, -y \rangle^-]$. From (11), when $C = 0$, then clearly $A \geq B$.

($\Leftarrow$). Here, we prove that $A \geq B$ implies $C = 0$. So, by (11), we have that

$$A \geq B + C. \tag{14}$$

Since $A \geq B$ then, by (14), we conclude that

$$0 \geq C. \tag{15}$$

By Theorem 2.6,

$$C < 0 \text{ implies } A < B.$$

This implies, by contrapositive argument, that

$$A \geq B \text{ implies } C \geq 0. \tag{16}$$

Hence, by (15) and (16), we obtain

$$A \geq B \text{ implies } C = 0.$$

Therefore, the proof is complete.

Theorem 2.8 (Convex type inequality). Let X be a real quasi bi-inner product space and let $\lambda \in (0, 1)$. Then for all $x, y \in X$

$$\begin{aligned} \|\lambda x + (1 - \lambda)y\|^2 &\geq \lambda\|x\|^2 + (1 - \lambda) [\lambda\|y\|^2 + (1 - \lambda)\|y\|^2] \\ &+ 2\lambda(1 - \lambda)[\langle x, y \rangle^- + \langle x, -y \rangle^-] - \lambda(1 - \lambda)\|x - y\|^2. \end{aligned} \quad (17)$$

Proof. The proof of (17) is done by expanding the left hand side to obtain the right hand side.

Theorem 2.9 (Pythagorean type inequalities). Let X be a real quasi bi-inner product space. Then for all $x, y \in X$,

$$\|x + y\|^2 \geq \|x\|^2 + 2\langle x, y \rangle^- + \|y\|^2, \quad (18)$$

$$\|x - y\|^2 \geq \|x\|^2 + 2\langle x, -y \rangle^- + \|y\|^2. \quad (19)$$

Proof. The proof of (18) and (19) are done by expanding the left hand sides to obtain the right hand sides.

Theorem 2.10 (Polarization type inequalities). Let X be a real quasi bi-inner product space. Then for all $x, y \in X$,

$$\langle x, y \rangle^- \leq \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + \|y\|^2 - \|y\|^2), \quad (20)$$

$$\langle x, -y \rangle^- \leq \frac{1}{4}(\|x - y\|^2 - \|x + y\|^2 - \|y\|^2 + \|y\|^2). \quad (21)$$

Proof. By (12), subtracting (19) from (18), we get (20). Similarly, when (18) is taken from (19) and (12) is applied, we also have (21), as desired.

Theorem 2.11 (Jordan-Von Neumann type theorem).

Let X be a real asymmetric normed space. Then the functions $\|\cdot\|$ and $|\cdot|$ are given by, respectively, $\langle \cdot, \cdot \rangle^+$ and $\langle \cdot, \cdot \rangle^-$ with $\langle x, y \rangle^- + \langle x, -y \rangle^- = 0$, for all $x, y \in X$, if and only if

$$\|x + y\|^2 + \|x - y\|^2 \geq 2(\|x\|^2 + 2^{-1}(\|y\|^2 + \|y\|^2)). \quad (22)$$

Proof. ($\implies$). By Theorem 2.7, we get the necessary condition. ($\impliedby$). Since $\langle x, y \rangle^- = -\langle x, -y \rangle^-$, then clearly condition (Q2), (Q3)

and (Q5) hold trivially. For (Q1), we have, by (21) and (22)

$$\begin{aligned}\langle x, x \rangle^- &= -\langle x, -x \rangle^-, \\ &\geq \frac{1}{4}(-\|x - x\|^2 + \|x + x\|^2 + \|x\|^2 - \|x\|^2), \\ &\geq \frac{1}{2}(-\|x - x\|^2 + \|x\|^2 + \|x\|^2) = \|x\|^2 \geq 0.\end{aligned}$$

Next, we show condition (Q4). So, by (20)

$$\begin{aligned}\langle x + y, -z \rangle^- &\leq \frac{1}{4}(\|x + y - z\|^2 - \|x + y + z\|^2 - \|z\|^2 + \|z\|^2), \\ &\leq \frac{1}{4}(\|x + y - z\|^2 - \|x + y + z\|^2 + 2\|z\|^2 + \|z\|^2).\end{aligned}\tag{23}$$

But, by (22), we have that

$$-\|x + y + z\|^2 \leq \|x + y - z\|^2 - 2\|x + y\|^2 - \|z\|^2.\tag{24}$$

Substituting (24) into (23), we obtain

$$\langle x + y, -z \rangle^- \leq \frac{1}{4}(2\|x + y - z\|^2 - 2\|x + y\|^2 + 2\|z\|^2).\tag{25}$$

Interchanging the role of $x + y$ and $x + y - z$ in (25), we get

$$\begin{aligned}\langle x + y, -z \rangle^- &\leq \langle x + y, -z \rangle^- + \langle -z, -z \rangle^- \leq \langle x + y - z, -z \rangle^- \\ &\leq \frac{1}{4}(2\|x + y\|^2 - 2\|x + y - z\|^2 + 2\|z\|^2).\end{aligned}\tag{26}$$

Replacing $x + y - z$ by $x + y + z$ in (26), we have

$$\begin{aligned}\langle x + y, -z \rangle^- &\leq \frac{1}{4}(2\|x + y\|^2 - 2\|x + y + z\|^2 + 2\|z\|^2) \\ &\leq \frac{1}{4}(2\langle x + y, x + y \rangle^- - 2\langle x + y, x + y \rangle^- - 4\langle x + y, z \rangle^- \\ &\quad - 2\langle z, z \rangle^- + 2\langle z, z \rangle^-) = -\langle x, z \rangle^- - \langle y, z \rangle^-.\end{aligned}$$

This implies, by the hypothesis of the theorem, that

$$\langle x + y, z \rangle^- \geq \langle x, z \rangle^- + \langle y, z \rangle^-,$$

as in (Q4). In a similar way, we clearly see that, $\langle \cdot, \cdot \rangle^+$ and $\langle \cdot, \cdot \rangle^-$ satisfy the axioms of quasi inner product and conjugate quasi inner product, respectively. Hence, the result holds, as desired.

2.5 Biorthonormal sets

Definition 2.6 Two vectors x and y in a real quasi bi -inner product space X are said to be bi -orthogonal (written as $x \perp_b y$ or

$y \perp_b x$) if $\langle x, y \rangle^+ = \langle x, y \rangle^- = 0$. If M is a subset of X , then we write $x \perp_b M$ if $x \perp_b y$ for every $y \in M$.

Definition 2.7 Let $d \geq 0$. A set S in a real quasi bi -inner product space X is called a bi -orthogonal set if $\langle x, y \rangle^+ = \langle x, y \rangle^- = 0$ for every $x, y \in S$, $x \neq y$. The set S is called bi -orthonormal if it is a bi -orthogonal set and $\|x\| = 1$ and $|x| = d$ for each $x \in S$.

Theorem 2.13 (Bessel's inequality type theorem). If $\{u_i\}_{i=1}^\infty$ is a bi -orthonormal set in a real quasi bi -inner product space X , then for arbitrary $x \in X$,

$$\|x\|^2 \geq \begin{cases} \sum_{i=1}^\infty |\langle x, u_i \rangle^-|^2, & \text{if } \langle x, u_i \rangle^- \leq 0 \\ 2 \sum_{i=1}^\infty \langle x, u_i \rangle^- \langle x, -u_i \rangle^-, & \text{if } \langle x, u_i \rangle^- > 0 \end{cases} \quad (27)$$

and, for some $d \geq 0$

$$|x|^2 \geq \begin{cases} d \sum_{i=1}^\infty |\langle x, u_i \rangle^+|^2, & \text{if } \langle x, u_i \rangle^+ \leq 0 \\ 2 \sum_{i=1}^\infty \langle x, u_i \rangle^+ \langle x, -u_i \rangle^+, & \text{if } \langle x, u_i \rangle^+ > 0. \end{cases} \quad (28)$$

Proof. Clearly,

$$\left\| x - \sum_{i=1}^N \langle x, u_i \rangle^- u_i \right\|^2 \geq 0,$$

so that, $\left\langle x - \sum_{i=1}^N \langle x, u_i \rangle^- u_i, x - \sum_{i=1}^N \langle x, u_i \rangle^- u_i \right\rangle^- \geq 0$, which implies that

$$\begin{aligned} & \left\langle x - \sum_{i=1}^N \langle x, u_i \rangle^+ u_i, x - \sum_{i=1}^N \langle x, u_i \rangle^+ u_i \right\rangle^- \\ & \geq \langle x, x \rangle^- + 2 \left\langle x, -\sum_{i=1}^N \langle x, u_i \rangle^+ u_i \right\rangle^- \\ & + \left\langle -\sum_{i=1}^N \langle x, u_i \rangle^+ u_i, -\sum_{i=1}^N \langle x, u_i \rangle^+ u_i \right\rangle^-. \end{aligned} \quad (29)$$

Interchanging the role of x and $x - \sum_{i=1}^N \langle x, u_i \rangle^+ u_i$ in (29), we get

$$\begin{aligned}
0 &\leq \left\langle x - \sum_{i=1}^N \langle x, u_i \rangle^- u_i, x - \sum_{i=1}^N \langle x, u_i \rangle^- u_i \right\rangle^- \\
&\leq \langle x, x \rangle^- - 2 \left\langle x - \sum_{i=1}^N \langle x, u_i \rangle^- u_i, - \sum_{i=1}^N \langle x, u_i \rangle^+ u_i \right\rangle^- \\
&\quad - \left\langle - \sum_{i=1}^N \langle x, u_i \rangle^- u_i, - \sum_{i=1}^N \langle x, u_i \rangle^+ u_i \right\rangle^- \\
&\leq \langle x, x \rangle^- - 2 \left\langle x, - \sum_{i=1}^N \langle x, u_i \rangle^- u_i \right\rangle^- \\
&\quad - 2 \left\langle - \sum_{i=1}^N \langle x, u_i \rangle^- u_i, - \sum_{i=1}^N \langle x, u_i \rangle^+ u_i \right\rangle^- \\
&\quad - \left\langle - \sum_{i=1}^N \langle x, u_i \rangle^- u_i, - \sum_{i=1}^N \langle x, u_i \rangle^- u_i \right\rangle^- . \tag{30}
\end{aligned}$$

If $\langle x, u_i \rangle^- \leq 0$, then from (30)

$$0 \leq \|x\|^2 - \sum_{i=1}^N |\langle x, u_i \rangle^-|^2. \tag{31}$$

Else if, $\langle x, u_i \rangle^- > 0$, then by (30)

$$0 \leq \langle x, x \rangle^- - 2 \left\langle x, - \sum_{i=1}^N \langle x, u_i \rangle^+ u_i \right\rangle^- . \tag{32}$$

So, by (32), we arrive at

$$0 \leq \|x\|^2 - 2 \sum_{i=1}^N \langle x, u_i \rangle^- \langle x, -u_i \rangle^- . \tag{33}$$

Hence, by (31), (33) and when setting N to go to infinity, we get (27) and in a similar way, we also obtain (28), respectively, as requested.

2.6 Projection type theorems and its applications

In [2], the balls with respect to $\|\cdot\|$, denoted by $B^{\|\cdot\|}$, are called forward balls, while the balls with respect to $|\cdot|$, given by $B^{|\cdot|}$, are called backward balls.

Definition 2.8 Let H^b be a real quasi *bi*Hilbert space and let C be a subset of H^b . A point $x \in H^b$ is said to be an $\|\cdot\|$ -limit ($|\cdot|$ -limit) point of C if every open $\|\cdot\|$ -ball ($|\cdot|$ -ball) centred at x contains at least one point of C different from x . It is therefore called a bilimit point of C if it is both $\|\cdot\|$ -limit and $|\cdot|$ -limit point of C .

Definition 2.9 A subset C of H^b is said to be $\|\cdot\|$ -bounded ($|\cdot|$ -bounded) if there exists a constant $M \geq 0$ ($M' \geq 0$) such that $\|x\| \leq M$ ($|x| \leq M'$), for all $x \in C$. Then C is binorm bounded if it is both $\|\cdot\|$ -bounded and $|\cdot|$ -bounded.

Definition 2.10 Suppose that $C \subset \mathbb{R}$ is binorm bounded with lower $\|\cdot\|$ -bound $\alpha_0 \in \mathbb{R}$ and lower $|\cdot|$ -bound $\beta_0 \in \mathbb{R}$ such that

- (1) $\gamma_0 = \min\{\alpha_0, \beta_0\} \leq x, \forall x \in C$
- (2) $\exists x_0 \in C$, for all $\epsilon > 0$, $\gamma_0 \leq x_0 < \gamma_0 + \epsilon$.

Then γ_0 is called binorm infimum and it is denoted by $\inf_b$.

Definition 2.11 Let H^b be a real quasi *bi*Hilbert space and C be a subset of H^b . C is $\|\cdot\|$ -closed ($|\cdot|$ -closed) if for any sequence $\{x_n\} \in C$ which is $\|\cdot\|$ -convergent to x ($|\cdot|$ -convergent to x) then $x \in C$. It is therefore biclosed if it is both $\|\cdot\|$ -closed and $|\cdot|$ -closed.

Theorem 2.14 (Projection type theorem: existence and uniqueness of minimizing vector). Let X be a real quasi *bi*-inner product space, M a biclosed subspace of X , and $x, y \in X$ with $\langle x, y \rangle^- + \langle x, -y \rangle^- = 0$. Then, there exists a unique vector $x^* \in M$ such that

$$\|x - x^*\|^2 \leq \|x - y\|^2 \text{ (similarly, } |x - x^*|^2 \leq |x - y|^2) \quad \forall y \in M.$$

Furthermore, $x^* \in M$ is a unique minimizing vector if and only if $(x - x^*) \perp M$.

Proof. First, we prove the existence of a minimizing vector. Let $x \in M$, then by choosing $x^* = x$, we are done with the proof. But, if we assume $x \notin M$, then, by the definition of binorm infimum, we may assume $\{y_n\}_{n=1}^\infty$ to be a sequence of vectors in M such that

$$\|x - y_n\| \text{ } \|\cdot\| \text{-converges to } \|x - x^*\|, \text{ as } n \rightarrow \infty$$

and by Definition 2.3

$$|x - y_n| \text{ } |\cdot| \text{-converges to } |x - x^*| = \|x - x^*\|, \text{ as } n \rightarrow \infty. \quad (34)$$

By the hypothesis of the theorem, we write the parallelogram type law,

$$\begin{aligned} \|(y_n - x) + (x - y_k)\|^2 + \|(y_n - x) - (x - y_k)\|^2 \\ \geq 2 \|y_n - x\|^2 + \|x - y_k\|^2 + \|x - y_k\|^2, \end{aligned}$$

which implies

$$\begin{aligned} \|(y_n - x) + (x - y_k)\|^2 + \|(y_n - x) - (x - y_k)\|^2 \\ \geq \|y_n - x\|^2 + \|x - y_k\|^2. \end{aligned} \quad (35)$$

Replacing $y_n - x$ by $(y_n - x) + (x - y_k)$ and $x - y_k$ by $(y_n - x) - (x - y_k)$ in (35), we have by (19)

$$\begin{aligned} \|y_n - y_k\|^2 &\leq \|y_n - x\|^2 + \|x - y_k\|^2 - \|(y_n - x) - (x - y_k)\|^2 \\ &\leq \|y_n - x\|^2 + \|x - y_k\|^2 \\ &\quad - (\|y_n - x\|^2 + 2 \langle -(x - y_n), -(x - y_k) \rangle^- + \|x - y_k\|^2) \\ &\leq \|y_n - x\|^2 + \|x - y_k\|^2 - 2 \langle (x - y_n), (x - y_k) \rangle^+ \end{aligned}$$

and therefore

$$\|y_n - y_k\| \|\cdot\| \text{-converges to 0 as } n, k \rightarrow \infty \text{ (by (34))}.$$

Hence, since M is bicomplete, $\{y_n\}_{n=1}^\infty$ has a bilimit point $x^* \in M$ and so, by the uniqueness of the bilimit and the continuity of the asymmetric norm, we get

$$\lim_{n \rightarrow \infty} \|x - y_n\| = \left\| x - \lim_{n \rightarrow \infty} y_n \right\| = \|x - x^*\| \leq \|x - y\|, \quad \forall y \in M.$$

For the uniqueness, we consider the parallelogram type law,

$$\begin{aligned} \|(x^* - x) + (x - x')\|^2 + \|(x^* - x) - (x - x')\|^2 \\ \geq 2 \|x^* - x\|^2 + \|x - x'\|^2 + \|x - x'\|^2, \end{aligned}$$

which implies

$$\begin{aligned} \|(x^* - x) + (x - x')\|^2 + \|(x^* - x) - (x - x')\|^2 \\ \geq \|x^* - x\|^2 + \|x - x'\|^2. \end{aligned} \quad (36)$$

Now, we assume there exists $x' \in M$ a minimizing vector such that $x^* \neq x'$. Interchanging the role of $x^* - x$ and $(x^* - x) + (x - x')$ and of $x - x'$ and $(x^* - x) + (x - x')$ in (36), we have by (19)

$$\begin{aligned}
0 &< \|x^* - x'\|^2 \leq \|x^* - x\|^2 + \|x - x'\|^2 - \|(x^* - x) - (x - x')\|^2 \\
&\leq \|x^* - x\|^2 + \|x - x'\|^2 \\
&\quad - \left(\|x^* - x\|^2 + 2\langle -(x - x^*), -(x - x') \rangle^- + \|x - x'\|^2 \right) \\
&\leq \|x^* - x\|^2 + \|x - x'\|^2 - 2\langle x - x^*, x - x' \rangle^+.
\end{aligned}$$

Since x' is a minimizing vector, then $\|x - x'\| = \|x - x^*\|$ and so by (34) we conclude that

$$0 < \|x^* - x'\| = 0.$$

This is a contradiction. Hence, the minimizing vector is unique.

Finally, we prove the last part of the Theorem.

($\Rightarrow$). On contrary, let assume that there exists $0 \neq x_0 \in M$ which is not *bi*-orthogonal to $(x - x^*)$. Without lose of generality, let $\|x_0\| = 1$, $\langle x - x^*, x_0 \rangle^- = -\delta$ and $\langle x^* - x, -x_0 \rangle^- = 3\delta$, for some $\delta > 0$. Define a vector $x_\delta = x^* + \delta x_0$. Then, by (12) and the Cauchy-Schwartz type inequality

$$\begin{aligned}
\|x - x_\delta\|^2 &= \|x - x^* - \delta x_0\|^2 \\
&= \langle x - x^* - \delta x_0, x - x^* - \delta x_0 \rangle^- \\
&\leq -\langle x - x^* - \delta x_0, x^* - x + \delta x_0 \rangle^- \\
&= -\langle x - x^*, x^* - x \rangle^- - \delta \langle x - x^*, x_0 \rangle^- \\
&\quad - \delta \langle -x_0, x^* - x \rangle^- - \delta^2 \langle -x_0, x_0 \rangle^- \\
&\leq \|x - x^*\| \|x^* - x\| + \delta^2 - 3\delta^2 + \delta^2 \text{ (since } \langle x_0, x_0 \rangle^- = -\langle -x_0, x_0 \rangle^-) \\
&\leq \|x - x^*\|^2 - \delta^2 < \|x - x^*\|^2 \text{ (by (34))}.
\end{aligned}$$

This is a contradiction. Hence $(x - x^*) \perp M$.

($\Leftarrow$). Let $(x - x^*) \perp M$. For arbitrary $x^* \neq y \in M$, we have, for $x \in X$ and by the Pythagorean type inequality

$$\begin{aligned}
\|x - y\|^2 &= \|(x - x^*) + (x^* - y)\|^2 \\
&\geq \|x - x^*\|^2 + \|x^* - y\|^2
\end{aligned}$$

which implies that

$$\|x - y\| > \|x - x^*\|.$$

This shows that x^* is the minimizing vector. Hence, the proof is complete.

Theorem 2.15 (Direct sum decomposition type theorem). Let M be a biclosed subspace of a real quasi *bi*Hilbert space, H^b . Then $H^b = M \oplus M^\perp$.

Proof. By Projection type theorem, for any $x \in H^b$, there exists a unique vector $x^* \in M$ such that

$$\|x - x^*\| \leq \|x - y\| \quad \forall y \in M$$

and

$$z^* = x - x^* \in M^\perp.$$

Hence, we can write

$$x = x^* + (x - x^*) = x^* + z^*, \text{ where } z^* = x - x^*,$$

with $x^* \in M$ and $z^* \in M^\perp$. Next we show that $x^* + z^*$ is unique. Suppose that $x' + z'$ with $x' \in M$ and $z' \in M^\perp$ is another representation of x . Then,

$$x^* + z^* = x' + z' \text{ so that } x' - x^* + (z' - z^*) = 0.$$

But $(x' - x^*)$ and $(z' - z^*)$ are bi-orthogonal. Hence, by Pythagorean type theorem,

$$0 = \|(x' - x^*) + (z' - z^*)\|^2 \geq \|x' - x^*\|^2 + \|z' - z^*\|^2 \geq 0.$$

This implies, $\|x' - x^*\| = 0$ and $\|z' - z^*\| = 0$, establishing the uniqueness of the representation. Thus, the result holds, as desired.

Definition 2.12 Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be real asymmetric normed spaces, and let $A_b : X \rightarrow Y$ be a linear operator. Then A_b is said to be $(\|\cdot\|_1, \|\cdot\|_2)$ -bounded linear $((\|\cdot\|_1, \|\cdot\|_2)$ -bounded linear) if there exists some constant $K \geq 0$ ($K' \geq 0$) such that

$$\|A_b(x)\|_2 \leq K\|x\|_1 \text{ (resp., } \|A_b(x)\|_2 \leq K'\|x\|_1), \quad \forall x \in X.$$

It is therefore called binorm bounded linear operator if it is both $(\|\cdot\|_1, \|\cdot\|_2)$ -bounded linear and $(\|\cdot\|_1, \|\cdot\|_2)$ -bounded linear.

Definition 2.13 Let H^b be a real quasi *bi*Hilbert space. The set of all binorm bounded and linear functionals $f : H^b \rightarrow \mathbb{R}$, denoted by H^{b*} , is called quasi bidual space.

Theorem 2.16 Let H^b be a real quasi *bi*Hilbert space and let $\langle x, y \rangle^- + \langle x, -y \rangle^- = 0$, for all $x, y \in H^b$. Then $H^{b*} = H^b$.

Proof. We will divide the proof in to two steps.

Step one. We show that every $x \in H^b$ defines an element $f_x \in H^{b*}$ with $\|f_x\| = \|x\|$. Now, let $x \in H^b$ and let $y \in H^b$ be chosen arbitrarily. Define $f_x : H^b \rightarrow K \subset H^b$ by

$$f_x(y) = \langle y, x \rangle^-.$$

Observe that f_x is linear. Furthermore,

$$|f_x(y)| = |\langle y, x \rangle^-| \leq \|y\| \|x\|,$$

so that f_x is $\|\cdot\|$ -bounded. Therefore, $f_x \in H^{b*}$. Moreover, the last inequality above gives

$$\|f_x\| \leq \|x\|. \quad (37)$$

For $x = 0$, by (45), $f_x = 0$ and so we are done with the proof. But, suppose that $x \neq 0$, then we let $y = \frac{x}{\|x\|}$ with $\|y\| = 1$ and $f_x(y) = \|x\|$. This implies that

$$\|f_x(y)\| = \|x\| \leq \|f_x\| \|y\| = \|f_x\| \text{ so that } \|x\| \leq \|f_x\|. \quad (38)$$

By (45) and (50), we get $\|f_x\| = \|x\|$.

Step two. We prove that every $f \in H^{b*}$ defines a unique vector $x \in H^b$ with $\|f\| = \|x\|$. Hence H^b and H^{b*} are isometric. Now, let $f \in H^{b*}$. Define the kernel of f by

$$\text{Ker } f := \{u \in H^b : f(u) = 0\}.$$

Let $K := \text{Ker } f$. Clearly K is $\|\cdot\|$ -closed subspace of H^b . By Theorem 2.10, we can write $y \in H^b$ as $y = w + z$ where $w \in K$ and $z \perp K$. So, for $y \notin K$ and $y = w^* + z$, where $z \neq 0$ and $f(z) = \delta \neq 0$, let $y_1 = \frac{z}{\delta}$. Then $f(y_1) = 1$. Hence, for arbitrary $u \in H^b$, $u \notin K$, we get that

$$f(u) = \alpha \Rightarrow f(u) = \alpha f(y_1), \text{ that is, } f(u - \alpha y_1) = 0.$$

Let $u - \alpha y_1 = w'$. Hence, $u = w' + \alpha y_1$, $w' \in K$ and $\alpha y_1 \perp K$ such that

$$\begin{aligned} \langle u, y_1 \rangle^- &= \langle w' + \alpha y_1, y_1 \rangle^- \\ &\geq \langle w', y_1 \rangle^- + \langle \alpha y_1, y_1 \rangle^- = \begin{cases} \alpha \langle y_1, y_1 \rangle^-, & \text{if } \alpha \geq 0 \\ -\alpha \langle -y_1, y_1 \rangle^-, & \text{if } \alpha < 0. \end{cases} \end{aligned} \quad (39)$$

By (39) and the hypothesis of the theorem, we have

$$f(u) \leq \begin{cases} \left\langle u, \frac{y_1}{\|y_1\|^2} \right\rangle^-, & \text{if } \alpha \geq 0 \\ \left\langle u, \frac{y_1}{-\langle -y_1, y_1 \rangle^-} \right\rangle^- = \left\langle u, \frac{y_1}{\|y_1\|^2} \right\rangle^-, & \text{if } \alpha < 0. \end{cases}$$

Hence,

$$f(u) \leq \langle u, y_0 \rangle^- \text{ where } y_0 = \frac{y_1}{\|y_1\|^2}. \quad (40)$$

By (21), we have that

$$\langle u, -y_1 \rangle^- \leq \frac{1}{4}(\|u - y_1\|^2 - \|u + y_1\|^2 - \|y_1\|^2 + 2\|y_1\|^2). \quad (41)$$

But, by the hypothesis of the theorem, we get the parallelogram inequality,

$$\|u + y_1\|^2 + \|u - y_1\|^2 \geq 2\|u\|^2 + \|y_1\|^2 + \|y_1\|^2.$$

This implies that

$$\|y_1\|^2 + \|u + y_1\|^2 + \|u - y_1\|^2 \geq 2\|u\|^2 + \|y_1\|^2 \geq 2\|u\|^2,$$

and therefore

$$-\|u + y_1\|^2 \leq \|u - y_1\|^2 - 2\|u\|^2 + \|y_1\|^2. \quad (42)$$

Substituting (42) into (41), we obtain

$$\langle u, -y_1 \rangle^- \leq \frac{1}{4}(2\|u - y_1\|^2 - 2\|u\|^2 + 2\|y_1\|^2). \quad (43)$$

Interchanging the role of u and $u - y_1$ in (43), we get

$$\langle u - y_1, -y_1 \rangle^- \leq \frac{1}{4}(2\|u\|^2 - 2\|u - y_1\|^2 + 2\|y_1\|^2),$$

so that

$$\langle u, -y_1 \rangle^- + \langle -y_1, -y_1 \rangle^- \leq \frac{1}{4}(2\|u\|^2 - 2\|u - y_1\|^2 + 2\|y_1\|^2),$$

and therefore

$$\begin{aligned} \langle u, -y_1 \rangle^- &\leq \frac{1}{4}(2\|u\|^2 - 2\|u - y_1\|^2 + 2\|y_1\|^2) \\ &= \frac{1}{4}(2\langle w' + \alpha y_1, w' + \alpha y_1 \rangle^- - 2\langle w' + \alpha y_1 - y_1, w' + \alpha y_1 - y_1 \rangle^- \\ &\quad + 2\langle -y_1, -y_1 \rangle^-) \leq \frac{1}{4}(-4\langle w' + \alpha y_1, -y_1 \rangle^-) \\ &= -\langle w' + \alpha y_1, -y_1 \rangle^- \leq -\langle w', -y_1 \rangle^- - \langle \alpha y_1, -y_1 \rangle^- \\ &= -\langle \alpha y_1, -y_1 \rangle^-. \end{aligned}$$

By the hypothesis of the theorem and interchanging the role of y_1 and $-y_1$ in the last inequality above, we arrive at

$$f(u) \geq \begin{cases} \left\langle u, \frac{y_1}{\|y_1\|^2} \right\rangle^-, & \text{if } \alpha < 0 \\ \left\langle u, \frac{y_1}{-\langle -y_1, y_1 \rangle^-} \right\rangle^- = \left\langle u, \frac{y_1}{\|y_1\|^2} \right\rangle^-, & \text{if } \alpha \geq 0. \end{cases}$$

Hence,

$$f(u) \geq \langle u, y_0 \rangle^- . \quad (44)$$

By (40) and (44), we conclude that

$$f(u) = \langle u, y_0 \rangle^- .$$

In addition, by *Step one* of the proof above, we have that $\|f\| = \|y_0\|$. Finally, we will show that y_0 is unique. Suppose that $f(y) = \langle y, y^* \rangle^-$ for all $y \in H^b$, then

$$\langle y, y_0 \rangle^- = \langle y, y^* \rangle^- \Rightarrow \langle y, y_0 - y^* \rangle^- = 0.$$

Take $y = y_0 - y^*$, then

$$\langle y_0 - y^*, y_0 - y^* \rangle^- = \|y_0 - y^*\|^2 = 0 \Rightarrow y_0 = y^*.$$

The proof is complete.

Theorem 2.17 (Riesz representation type theorem). Let H^b be a real quasi *bi*Hilbert space and H^{b*} its quasi bidual. Let $f \in H^{b*}$, then there exists a unique vector,

- (1) $y_0 \in H^b$ such that $f(x) = \langle x, y_0 \rangle^-$ for each $x \in H^b$
- (2) $\|f\| = \|y_0\|$.

Similarly,

- (3) $-f(x) = \langle x, y_0 \rangle^+$ for each $x \in H^b$
- (4) $\|f\| = \|y_0\|$.

Proof. This follows from the proof of Theorem 2.16.

2.7 Bimetric projection and its characterizations

Definition 2.14 Let H^b be a real quasi *bi*Hilbert space and C be a biclosed and convex subset of H^b . A mapping $P_C^b : H^b \rightarrow C$ defined by, for $x \in H^b$, $P_C^b x = x_0$, where x_0 is the unique vector in C satisfying

$$\|x - x_0\| \leq \|x - y\| \quad (\text{similarly, } \|x - x_0\| \leq \|x - y\|) \quad \forall y \in C,$$

is called a *bi*-metric projection from H^b onto C .

Theorem 2.18 Let C be a nonempty biclosed and convex subset of a real quasi *bi*Hilbert space H^b . Let $y \in H^b$ and $x_0 \in C$, then for all $u \in C$ and $x_0 = P_C^b y$, the following inequalities are equivalent:

- (1) $\|y - x_0\| \leq \|y - u\|$,
- (2) $\langle x_0 - y, u - x_0 \rangle^+ \geq 0$,

- (3) $\|y - x_0\| \leq \|y - u\|,$
 (4) $\langle x_0 - y, u - x_0 \rangle^- \geq 0.$

Proof. (1 $\Rightarrow$ 2). Let $x_0 = P_C^b y$ and $u \in C$. Since C is convex, then for $t \in (0, 1)$, $tu + (1 - t)x_0 \in C$, we can write

$$\begin{aligned} \|y - x_0\|^2 &\leq \|y - (tu + (1 - t)x_0)\|^2 \\ &= \|y - x_0 - t(u - x_0)\|^2. \end{aligned} \quad (45)$$

Expanding the right hand side of (45), we get

$$\begin{aligned} &\langle y - x_0 - t(u - x_0), y - x_0 - t(u - x_0) \rangle^- \\ &\geq \langle y - x_0, y - x_0 \rangle^- + \langle y - x_0, -t(u - x_0) \rangle^- \\ &\quad + \langle -t(u - x_0), y - x_0 \rangle^- + \langle -t(u - x_0), -t(u - x_0) \rangle^- \\ &= \langle y - x_0, y - x_0 \rangle^- + 2t \langle y - x_0, (x_0 - u) \rangle^- \\ &\quad + t^2 \langle (x_0 - u), (x_0 - u) \rangle^-. \end{aligned} \quad (46)$$

Interchanging the role of $y - x_0$ and $y - x_0 - t(u - x_0)$ in (46), we have

$$\begin{aligned} &\langle y - x_0 - t(u - x_0), y - x_0 - t(u - x_0) \rangle^- \\ &\leq \langle y - x_0, y - x_0 \rangle^- - 2t \langle y - x_0 - t(u - x_0), (x_0 - u) \rangle^- \\ &\quad - t^2 \langle (x_0 - u), (x_0 - u) \rangle^- \\ &\leq \langle y - x_0, y - x_0 \rangle^- - 2t \langle y - x_0, (x_0 - u) \rangle^- \\ &\quad - 2t^2 \langle (x_0 - u), (x_0 - u) \rangle^- - t^2 \langle (x_0 - u), (x_0 - u) \rangle^- \\ &\leq \langle y - x_0, y - x_0 \rangle^- + 2t \langle y - x_0, (x_0 - u) \rangle^- \\ &\quad + t^2 \langle (x_0 - u), (x_0 - u) \rangle^-. \end{aligned} \quad (47)$$

This implies that, by (45) and (47)

$$\langle y - x_0, (x_0 - u) \rangle^- + \frac{t}{2} \langle (x_0 - u), (x_0 - u) \rangle^- \geq 0,$$

and therefore

$$\langle y - x_0, x_0 - u \rangle^- \geq 0 \text{ (as } t \rightarrow 0).$$

(2 $\Rightarrow$ 3). Let $u \in C$, then by (12)

$$\begin{aligned} 0 &\leq \langle y - x_0, x_0 - u \rangle^- = \langle x_0 - y, u - x_0 \rangle^+ \\ &\leq -\langle y - x_0, u - y + y - x_0 \rangle^+ \\ &\leq -\langle y - x_0, x_0 - y \rangle^+ - \langle y - x_0, y - u \rangle^+. \end{aligned}$$

This implies

$$\langle y - x_0, y - x_0 \rangle^+ \leq -\langle y - x_0, y - u \rangle^+$$

so that

$$\|y - x_0\| \leq \|y - u\|.$$

The proof of $(3 \Rightarrow 4)$ and $(4 \Rightarrow 1)$ follow from the computation technique of the proof of $(1 \Rightarrow 2)$ and $(2 \Rightarrow 3)$ above, respectively. The proof is complete.

Theorem 2.19 Let C be a nonempty biclosed and convex subset of a real quasi *bi*Hilbert space H^b and A be a nonlinear map. The *bi*-variational inequality associated with A and C is defined as finding $x_0 \in C$, such that

$$\langle Ax_0, u - x_0 \rangle^- \geq 0, \forall u \in C$$

similarly

$$\langle Ax_0, u - x_0 \rangle^- \geq 0$$

if and only if

$$x_0 = P_C^b(I - \lambda A)(x_0), \forall u \in C, \lambda > 0.$$

Proof. Let

$$\langle Ax_0, u - x_0 \rangle^- \geq 0. \quad (48)$$

Now multiply both sides of (59) by $\lambda > 0$, we have

$$\langle \lambda A(x_0), u - x_0 \rangle^- \geq 0$$

equivalently

$$\langle x_0 - P_C^b(I - \lambda A)(x_0), u - x_0 \rangle^- \geq 0$$

equivalently

$$x_0 = P_C^b(I - \lambda A)(x_0) \text{ for } \lambda > 0 \text{ (by Theorem 2.18).}$$

Theorem 2.20 Let H^b be a real quasi *bi*Hilbert space and C be a biclosed and convex subset of H^b . Let a mapping $P_C^b : H^b \rightarrow C$ be a *bi*-metric projection. For $x \in H^b$, if $x_0 = P_C^b x$ then,

$$\begin{aligned} \|x - x_0\|^2 + \|y - x_0\|^2 &\leq \|x - y\|^2 \\ (\text{similarly, } \|x - x_0\|^2 + \|y - x_0\|^2 &\leq \|x - y\|^2) \quad \forall y \in C. \end{aligned} \quad (49)$$

Proof. To show (49), let $y, x_0 \in C$ and $\lambda \in (0, 1)$, then $\lambda y + (1 - \lambda)x_0 \in C$. Clearly

$$\|x - x_0\|^2 \leq \|x - x_0 - \lambda(y - x_0)\|^2. \quad (50)$$

Using the same computation technique as in Theorem 2.18, we arrive at

$$\langle x_0 - x, y - x_0 \rangle^+ \geq 0. \quad (51)$$

By adding a special zeros ' $y - y$ ' and ' $x - x$ ' to the first and second argument in (51), we get respectively

$$\langle x_0 - y, y - x_0 \rangle^+ + \langle y - x_0, y - x \rangle^+ \geq 0$$

and

$$\langle x_0 - x, y - x \rangle^+ + \langle x_0 - x, x - x_0 \rangle^+ \geq 0.$$

This implies by (12) that

$$-\langle y - x_0, y - x_0 \rangle^+ + \langle y - x_0, y - x \rangle^+ \geq 0 \quad (52)$$

and

$$-\langle x - x_0, x - x_0 \rangle^+ + \langle x_0 - x, y - x \rangle^+ \geq 0. \quad (53)$$

Adding (52) and (53), we arrive at (49), as desired.

Theorem 2.21 For $x, y \in H^b$ and P_C^b be a bi-metric projection from H^b onto a biclosed and convex subset of H^b , C , we have

$$\|P_C^b x - P_C^b y\| \leq \|y - x\| + \|P_C^b x - x\| + \|y - P_C^b y\|,$$

similarly

$$\|P_C^b x - P_C^b y\| \leq \|y - x\| + \|P_C^b x - x\| + \|y - P_C^b y\|.$$

Proof. Let $x, y \in H^b$, then by definition

$$\|x - P_C^b x\| \leq \|x - u\| \text{ for all } u \in C. \quad (54)$$

For $\lambda \in (0, 1)$, since C is convex, then

$$\lambda P_C^b y + (1 - \lambda)P_C^b x \in C, \quad P_C^b x, P_C^b y \in C. \quad (55)$$

Substituting u with (55) in (54), we get

$$\|x - P_C^b x\|^2 \leq \|x - P_C^b x - \lambda(P_C^b y - P_C^b x)\|^2. \quad (56)$$

Expanding the right hand side of (56), we get

$$\langle x - P_C^b x, P_C^b x - P_C^b y \rangle^- \geq 0 \text{ as } \lambda \rightarrow 0,$$

so that

$$\langle P_C^b x - x, P_C^b y - P_C^b x \rangle^+ \geq 0. \quad (57)$$

From (57), interchanging the roles of x and y , we have

$$\langle P_C^b y - y, P_C^b x - P_C^b y \rangle^+ \geq 0. \quad (58)$$

Since, by (12), that

$$\langle P_C^b x - x, P_C^b x - P_C^b y \rangle^+ + \langle P_C^b x - x, P_C^b y - P_C^b x \rangle^+ \leq 0$$

then, by (57) and from the last inequality above, we get

$$\langle P_C^b x - x, P_C^b x - P_C^b y \rangle^+ \leq 0. \quad (59)$$

Similarly, when

$$\langle y - P_C^b y, P_C^b x - P_C^b y \rangle^+ + \langle P_C^b y - y, P_C^b x - P_C^b y \rangle^+ \leq 0$$

then, by (56) and from the last inequality above, we have

$$\langle y - P_C^b y, P_C^b x - P_C^b y \rangle^+ \leq 0. \quad (60)$$

Combining (59) and (60), we arrive at

$$\langle y - P_C^b y, P_C^b x - P_C^b y \rangle^+ + \langle P_C^b x - x, P_C^b x - P_C^b y \rangle^+ \leq 0. \quad (61)$$

But,

$$\begin{aligned} \langle (y - x) + (P_C^b x - P_C^b y), P_C^b x - P_C^b y \rangle^+ &\geq \langle y - P_C^b y, P_C^b x - P_C^b y \rangle^+ \\ &\quad + \langle P_C^b x - x, P_C^b x - P_C^b y \rangle^+. \end{aligned} \quad (62)$$

Interchanging the role of $y - P_C^b y$ and $(y - x) + (P_C^b x - P_C^b y)$, and of $P_C^b x - x$ and $(y - x) + (P_C^b x - P_C^b y)$ in (62), respectively, we have

$$\begin{aligned} \langle y - P_C^b y, P_C^b x - P_C^b y \rangle^+ &\geq \langle (y - x) + (P_C^b x - P_C^b y), P_C^b x - P_C^b y \rangle^+ \\ &\quad + \langle P_C^b x - x, P_C^b x - P_C^b y \rangle^+ \end{aligned} \quad (63)$$

and

$$\begin{aligned} \langle P_C^b x - x, P_C^b x - P_C^b y \rangle^+ &\geq \langle y - P_C^b y, P_C^b x - P_C^b y \rangle^+ \\ &\quad + \langle (y - x) + (P_C^b x - P_C^b y), P_C^b x - P_C^b y \rangle^+. \end{aligned} \quad (64)$$

By (61) and (63), we get

$$\begin{aligned} &\langle (y - x) + (P_C^b x - P_C^b y), P_C^b x - P_C^b y \rangle^+ \\ &\quad + 2 \langle P_C^b x - x, P_C^b x - P_C^b y \rangle^+ \leq 0, \end{aligned}$$

which implies

$$\begin{aligned} \langle P_C^b x - P_C^b y, P_C^b x - P_C^b y \rangle^+ &\leq -\langle (y - x), P_C^b x - P_C^b y \rangle^+ \\ &\quad - 2 \langle P_C^b x - x, P_C^b x - P_C^b y \rangle^+ \\ &\leq |\langle (y - x), P_C^b x - P_C^b y \rangle^+| + 2|\langle P_C^b x - x, P_C^b x - P_C^b y \rangle^+| \end{aligned}$$

and therefore

$$\|P_C^b x - P_C^b y\| \leq \|y - x\| + 2\|P_C^b x - x\|. \quad (65)$$

Similarly, by (61) and (64), we get

$$\|P_C^b x - P_C^b y\| \leq \|y - x\| + 2\|y - P_C^b y\|. \quad (66)$$

Combining (65) and (66), we conclude that

$$\|P_C^b x - P_C^b y\| \leq \|y - x\| + \|P_C^b x - x\| + \|y - P_C^b y\|,$$

as required.

2.8 Adjoint operator type on real quasi *bi*Hilbert spaces

Definition 2.15 Let H^b be a real quasi *bi*Hilbert space. Let $A_b : H^b \rightarrow H^b$ be a binorm bounded linear operator. We define a map

$$A_b^* : H^b \rightarrow H^b$$

by

$$\langle A_b^* y, x \rangle^+ = \langle A_b x, y \rangle^+ \text{ (similarly } \langle A_b^* y, x \rangle^- = \langle A_b x, y \rangle^-)$$

for all $x, y \in H^b$. The map A_b^* is called the adjoint operator type of binorm bounded linear operator, A_b .

Theorem 2.22 Let H^b be a real quasi *bi*Hilbert space. Let $A_b : H^b \rightarrow H^b$ be (binorm) bounded linear maps with adjoint operators A_b^* respectively. Then,

- (1) $\|A_b\| = \|A_b^*\|$;
- (2) $\|A_b^* A_b\| = \|A_b\|^2$.

Similarly

- (1) $|A_b| = |A_b^*|$;
- (2) $|A_b^* A_b| = |A_b|^2$.

Proof. This follows from the technique of proof in [1].

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